Finding an Analytical Solution for the Generalized Fermat Point

Is an analytical solution possible for the generalized Fermat point?

An Extended Essay in Mathematics

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IB Session May 2021

Word Count: 3846

Table of Contents

1.0 Introduction

In a world where systems are getting ever more complex, the efficient usage and allocation of resources becomes important to maximize consumer and resource efficiency. In the 1600s, de Fermat proposed a problem to Evangelista Torricelli where the objective was to find a point (known as the Fermat Point) that minimizes the sum of distances from it to three arbitrarily chosen points. This simple problem, known as the Fermat Torricelli problem, unwittingly spawned a rich library of literature with this idea of minimizing distances, or costs. A class of problems known as *facility location problems* investigates the optimal placement of facilities to minimize some sort of metric (Litoff, 2015). Specifically, the *k-medians* problem investigates optimal sites for placing facilities, or medians, such that the sum of distances from each client (an arbitrary set of points) to the nearest median is minimized (Durocher, 2006). The k-median problem in two dimensions remain NP-hard to compute (Karive & Hakimi, 1979), and it is unknown whether the problem with a fixed number of medians (like the generalized Fermat-Torricelli problem) is in the class NP as no algorithm has been found that can compute the exact location of even 1 median (Durocher, 2006). As all the literature on the generalized Fermat-Torricelli problem has been on finding more efficient approximation schemes, it is worthwhile to attempt an exact solution for this problem. Thus, I have been led to my research question, **is an analytical solution possible for the generalized Fermat-Torricelli Problem?**

1.1 Scope and Methodology

As the facility location problems are typically investigated at the graduate or post-graduate level, as a high school student, the mathematics required to understand the extensions of these problems are far beyond my reach. Therefore, to reduce the difficulty of the math involved but still provide a non-trivial investigation into the topic, this paper will focus on finding the cartesian coordinates of a point in a 2-dimensional Euclidean plane that minimizes the sum of Euclidean distances to a 2-dimensional set of arbitrary distributed k points. To build up to the focus of this paper, I will start from the simplest case of facility location problem, the Fermat Point. First asked in the 1600s by Italian mathematician de Fermat, the Fermat Point is one such that the sum of its distances to the 3 vertices of an arbitrary triangle is minimized (de Fermat, 1643). I will discuss 2 distinct approaches to the problem—geometrically and algebraically. Next, I will expand my scope to include four vertices, and discuss if the approaches outlined in section 2 can apply to this case. Lastly, I will extend this discussion even further to five points and beyond.

1.2 Definitions and Terminology

Durocher (2006) notes that there are numerous names assigned to the variants of facility location problems.[1] The solution of the Fermat-Torricelli Problem is known as the Fermat point. Hence, for the purposes of this paper, the point k that minimizes the sum of Euclidean distances from it to an arbitrary, not necessarily unique, number of cartesian points P in two dimensions will be referred to as *Fermat point for P vertices*. For example, the point that minimizes the sum of distances from itself to four arbitrary points is called the Fermat point for 4 vertices. However, the Fermat point for P vertices will sometimes be referred to simply as the Fermat point for convenience.

2.0 The Fermat-Torricelli Problem

There are many different solutions to the Fermat-Torricelli Problem, but I have chosen 2 that characterize vastly distinct approaches using the different disciplines of math.

A formal definition for the Fermat Point can be defined as follows: given 3 fixed points A, B, and C, the Fermat Point F minimizes the function:

$$d = |\mathrm{AF}| + |\mathrm{BF}| + |\mathrm{CF}| \quad \text{(Fig. 1)}$$

[1] This can be found in section 2.4.2

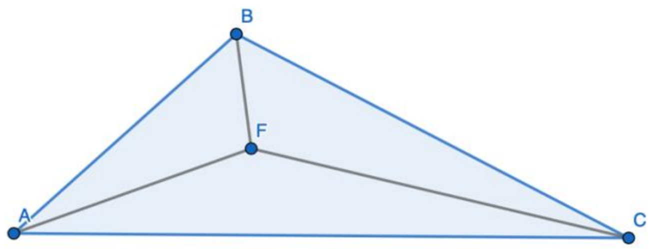

Figure 1: visualization of the Fermat point

2.1 Physical Approaches

It must first be noted that there are ways that one can ascertain the Fermat Point using physical devices. One of the most popular methods is through utilizing soap films and their property of minimizing free energy by minimizing surface area.

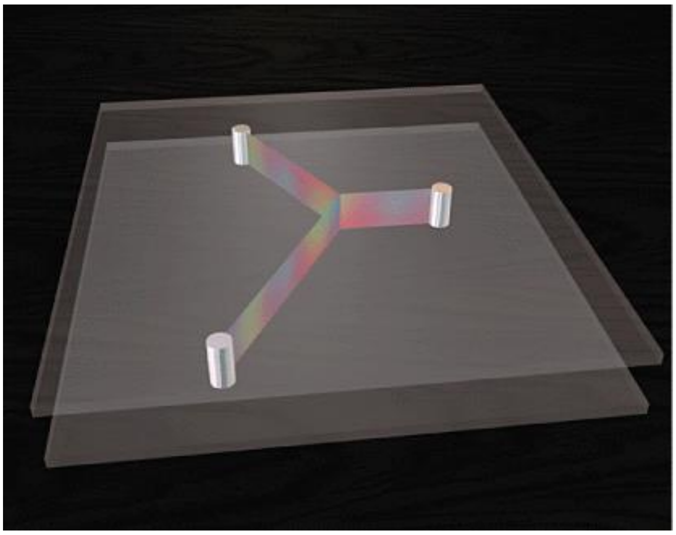

Figure 2: Soap bubbles naturally create Fermat points. Obtained from Glassner (2000)

2.2 Simpson's Geometric Solution

The solution for the Fermat Point was initially found by Italian mathematician Evangelista Torricelli, to whom the problem was directed to. There are many other geometric constructions for the Fermat Point (Bogolmony, n.d.), but Simpson's construction[2] will be presented as I consider it the most elegant proof. Similar proofs were presented by Park and Flores (2015), Bogolmony (n.d.), and more, but the following is my own interpretation. We distinguish between two cases where the arbitrary triangle has all angles $< 120^\circ$ or one angle $\geq 120^\circ$ for reasons that will become apparent soon.

2.2.1 <120˚ Case

Construct an arbitrary triangle ABC with angles less than 120˚ (The case of 120˚ will be examined later). Consider an arbitrary point F and connect it with vertices A, B, and C. In this set up, AF + BF + CF is the distance metric. Rotate ΔABF 60˚ around B and label the new vertices A' and F' (Fig. 3).

[2] Bajaj, 1988

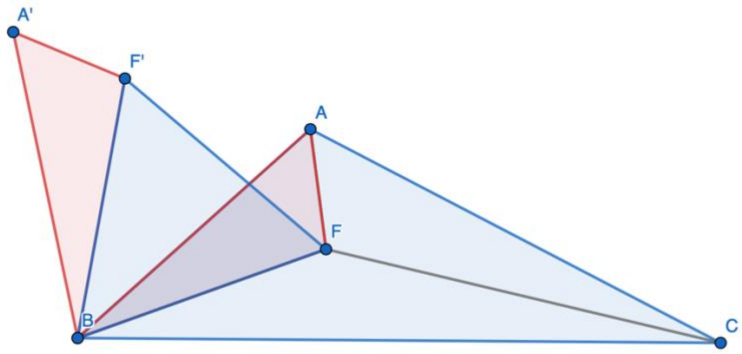

Figure 3: rotation of ΔABF

Notice that by construction, A'F' = AF, BF' = BF, and ∠F'BP = 60˚. Since ΔF'BF is isosceles and ∠F'BF = 60˚, ΔF'BF is an equilateral triangle. Thus, BF = F'F. Recall d = AF + BF + CF. This means that d also equals A'F' + F'F + CF. In other words, it is the distance of the path A'F'PC (Park and Flores, 2015). Notice, however, that ||A'C|| is minimized when A'F'PC is a straight line, since A' is a fixed point. Therefore, F lies on the line A'C. It should also be noted that ΔA'BA is equilateral since A'B = AB and ∠A'BA = 60˚ by construction (Fig. 3).

The same argument can be made rotating ΔBCP around C and ΔACP around A. Since each of the lines denote the minimum distance, and the Fermat Point lies on such a line, the Fermat Point must lie on all three lines. Therefore, they concur. Consequently, **the**

Fermat Point is located at the intersection of the lines connecting the vertices of triangle ABC to the vertices of its external equilateral triangles (Fig. 4).[3]

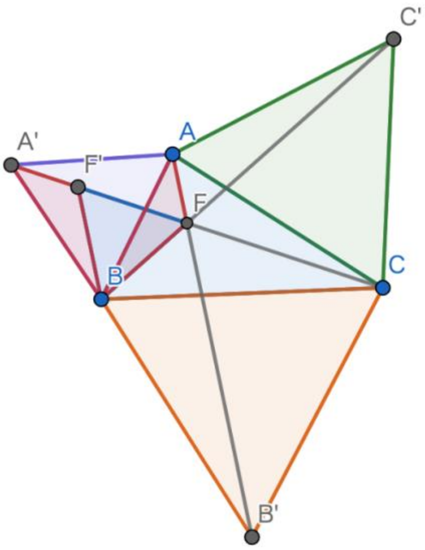

Figure 4: The Fermat Point lies at the intersection of A'C, BC', and AB'.

But what if one of the angles of the original triangle is greater or equal to 120˚? Using the compass and ruler method discussed above, the Fermat Point would turn out to be outside of the triangle (Fig. 5). The solution will be briefly addressed in the next subsection.

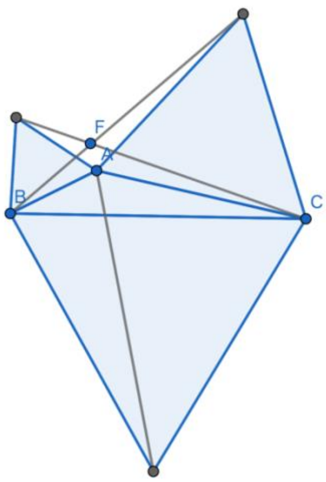

Figure 5: F lies outside of the triangle.

[3] Notice that since ∠BFF' = 60˚ by construction, ∠BFC = 180 – 60 = 120˚ and ∠AFB = 180 – 60 = 120˚. Thus, ∠AFC = 360 – 120 – 120 = 120˚. Hence, a property of the Fermat Point is that it forms 120˚ angles with the vertices of the triangle.

2.2.2 The ≥ 120˚ case

For the 120˚ case, the constructed Fermat Point will lie on the vertex with the 120˚ angle (Fig. 6). For angles greater than 120˚, it can be seen that placing the Fermat Point on the vertex with the >120˚ angle gives a lower total distance than where the compass and ruler construction dictates. And placing the point on said vertex turns out to be precisely the Fermat Point for angles greater than 120˚ (Park and Flores, 2015).[4]

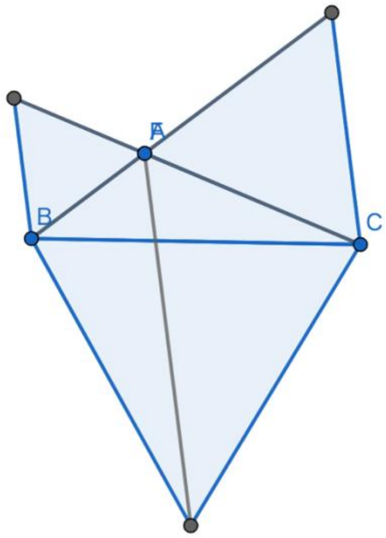

Figure 6: F converges to A

2.2.3 Confirmation of 2.2 Using Viviani's Theorem

Park and Flores (2015) give a proof for the Fermat Point using Viviani's Theorem. First, make equilateral triangle DEG by drawing perpendicular lines from AF, BF, and CF where F is the supposed Fermat Point found in section 2.2 (Fig. 7). Select an arbitrary point F' and draw A' F', B'F', C'F' such that these three lines are perpendicular to each corresponding side of ΔDEF (Fig. 7). Notice that A'F' ≤ AF', B'F' ≤ BF', and C'F' ≤ CF' as AF' is the hypotenuse of ΔA6'F' while A'F' is one of the side lengths, etc. Adding up the terms gives

[4] A more detailed proof of this can be found in Park and Flores (2015).

A’F’ + B’F’ + C’F’ < AF’ + BF’ + CF’ (AA’F’, BB’F’, CC’F’ will always form right triangles with AF’, BF’, CF’ as the hypotenuses unless A’, B’, C’ = A, B, C).

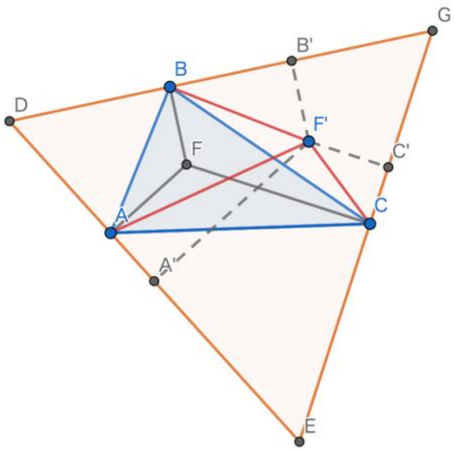

Figure 7: Viviani’s Theorem states that the sum of the lengths of the dotted lines equal the sum lengths of the solid gray lines.

Now, Viviani’s theorem states that the sum of the lengths of perpendicular line segments from any interior point of an equilateral triangle is equivalent to the length of its altitude (Bogolmony, n.d.). Thus,

AF + BF + CF = height of triangle = A’F’ + B’F’ + C’F’

Substituting into the previous inequality gives

AF + BF + CF < AF’ + BF’ + CF’

With equality iff F = F’. This leads to the conclusion that F is the Fermat Point. It is important to note that another consequence of this result is that the Fermat Point is unique.

2.4 Algebraic Solution to the Fermat Point

Hajja (1998) defines the fixed points P_1, P_2, and P_3 to have the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively, and the Fermat Point with coordinates (x, y) minimizes the function

$$f(x,y) = \sum_{i=1}^{3} \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

However, this notation is slightly confusing as will become apparent shortly, so I will call the coordinates of P_1, P_2, and P_3 to be (a_1, b_1), (a_2, b_2), and (a_3, b_3):

$$f(x,y) = \sum_{i=1}^{3} \sqrt{(x - a_i)^2 + (y - b_i)^2}$$

Using conventional optimization methods, we take the partial derivatives of the function and equate them to 0. This method is extensively employed by many such as Hajja, Simons, Escobar-Villagram, etc.

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{3} \frac{x - a_i}{\sqrt{(x - a_i)^2 + (y - b_i)^2}} = 0$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{3} \frac{y - b_i}{\sqrt{(x - a_i)^2 + (y - b_i)^2}} = 0$$

These two equations can then be multiplied by $\sum_{i=1}^{3}(x - a_i)^2 + (y - b_i)^2$, isolated for one of the like multiples of the two equations, set equal to each other, then regrouped to produce these three polynomial equations:

$$\frac{((x - a_2)(y - b_1) - (x - a_1)(y - b_2))^2}{((x - a_3)(y - b_1) - (x - a_1)(y - b_3))^2} = \frac{(x - a_2)^2 + (y - b_2)^2}{(x - a_3)^2 + (y - b_3)^2}$$

$$\frac{((x-a_1)(y-b_2)-(x-a_2)(y-b_1))^2}{((x-a_3)(y-b_2)-(x-a_2)(y-b_3))^2}=\frac{(x-a_1)^2+(y-b_1)^2}{(x-a_3)^2+(y-b_3)^2}$$
$$\frac{((x-a_1)(y-b_3)-(x-a_3)(y-b_1))^2}{((x-a_2)(y-b_3)-(x-a_3)(y-b_2))^2}=\frac{(x-a_1)^2+(y-b_1)^2}{(x-a_2)^2+(y-b_2)^2}$$

(see Appendix 2.4 for the algebraic steps)

Without loss of generality, the number of parameters can be reduced from 6 to 3 by setting point P_1 at the origin, and point P_2 on the x-axis, as the general solution can be obtained by simple transformations. Hence,

$$(a_1,\ b_1)=(0,0) \qquad\qquad (a_2,b_2)=(a_2\ ,0)$$

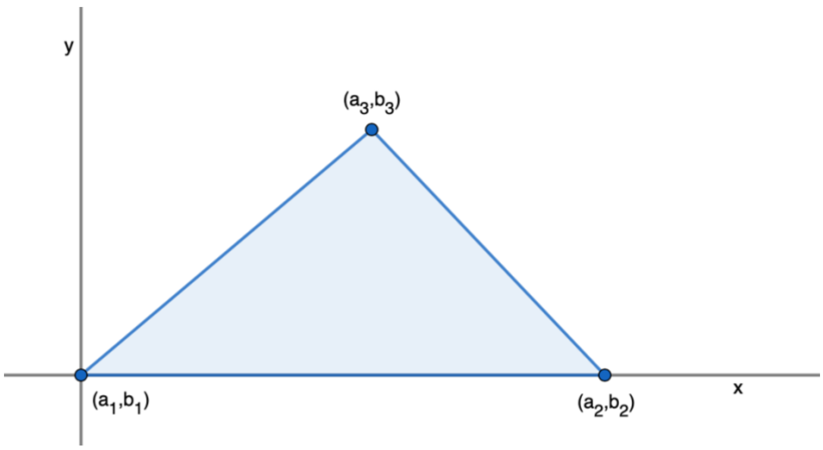

Figure 8: fixing a vertex at the origin and another on the x-axis

Through a lengthy and tedious process of algebraic manipulation (and with the aid of computer software), Escobar-Villagram et. al obtains the following equations for isolated x and y:

$$x=\frac{a_2(\sqrt{3}a_3+b_3)(\mathrm{a}_2+a_3+\sqrt{3}b_3)}{2\sqrt{3}(a_2{}^2-a_2a_3+a_3{}^2+b_3{}^2+\sqrt{3}a_2b_3)}$$
$$y=\frac{a_2(\sqrt{3}a_3+b_3)(\sqrt{3}(a_2-a_3)+b_3)}{2\sqrt{3}(\mathrm{a}_2^2-a_2a_3+a_3{}^2+b_3{}^2+\sqrt{3}a_2b_3)}$$

Escobar-Villagram claims that these formulas work for triangles with "interior angles no greater than 120°". The next section investigates the validity of this claim.

2.4.1 Confirmation through Random Sampling

This section will attempt to investigate if the algebraic solution holds true for randomly generated triangles. A Python program is written that randomly generated the coordinates a_2, a_3, and b_3, and checked if plugging the coordinates of the Fermat Point produced into the two partial derivative equations returns 0.[5] (See Appendix for the code).

The code can be found in the Appendix, but here is a brief overview of the workflow:

1. 3 random coordinates with integer values between 1 and 1 trillion are generated.
2. These coordinates were checked to see if they generated a valid triangle.
3. If not, random coordinates are generated again.
4. The coordinates are then plugged into the partial derivative equations.
5. A message will be raised if the equations do not equate to 0.
6. These steps are repeated 100 million times.

[5] Seen in section 2.4

For the < 120˚ and 120˚ cases, none of the samples generated violated the partial derivative equations. However, for the > 120˚ case, none of the samples generated satisfied the equations. This correlates to the findings in section 2.2.

3.0 The Case of Four Points

To tackle the problem of four points, the same methods used for the original three points can be considered. However, we once again distinguish the problem into two cases. We define the "floating case" as the case comprised of a convex quadrilateral and the "absorbed case" as the case dealing with concave quadrilaterals. The kinematic method does not seem to have any applications besides confirming what is already known (that the Fermat point is indeed what it claims to be) and thus will not be discussed in future sections.

3.1 Geometric method

Fagnano (1775) found the geometric solution for the Fermat Point to be the intersection of diagonals of the quadrilateral if it were convex (Fig. 9), and the concave vertex itself if the quadrilateral was concave (Fig. 10). The former case is readily proven with a simple application of the triangle inequality.

Let PQRS be any arbitrary quadrilateral in $\mathbb{R}^2$. Let U be a point such that U lies on the intersection of the diagonals of $PQRS$ (Fig. 9). Let X be an arbitrary point. By the triangle inequality,

$$PU + UR = PR \leq PX + XR$$

and

$$SU + UQ = SQ \leq SX + XQ$$

Adding the two inequalities together yields

$$PU + UR + SU + UQ \leq PX + XR + SX + XQ,$$

With equality iff $U = X$.

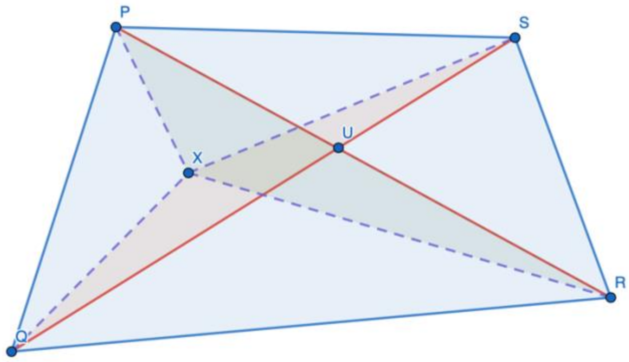

Figure 9: Visual representation of the above proof.

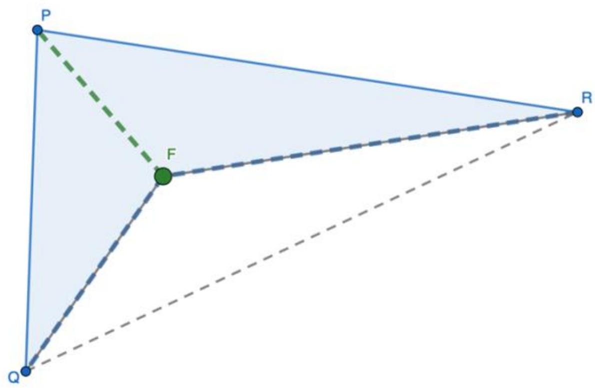

Figure 10: The Fermat point of a concave quadrilateral is the concave vertex.

For the concave case, we differentiate between the case where X is outside of PQR and when X is in PQR.

Case 1: X is outside of PQR (Fig. 11)

Theorem: For any arbitrary point X, there is always a point Y whose sum of distances is less than that of X.

Proof:

Firstly, if X is outside of PQR, there is a line that separates X from PQR. Find the orthogonal projection of X on the line and label it Y. By cosine law,

$$XQ^2 = XY^2 + YQ^2 - 2(XY)(YQ)cos\theta$$

Since $\theta \geq 90^{\circ}$ by construction, $-2(XY)(YQ)cos\theta \geq 0$ and $YQ^2 > 0$.

Therefore,

$$XQ^2 > YQ^2$$

$$XQ > YQ$$

The same argument can be made for XP, XR, and XS. Thus,

$$XP + XQ + XR + XS > YP + YQ + YR + YS$$

Consequently, X has to be inside PQR.

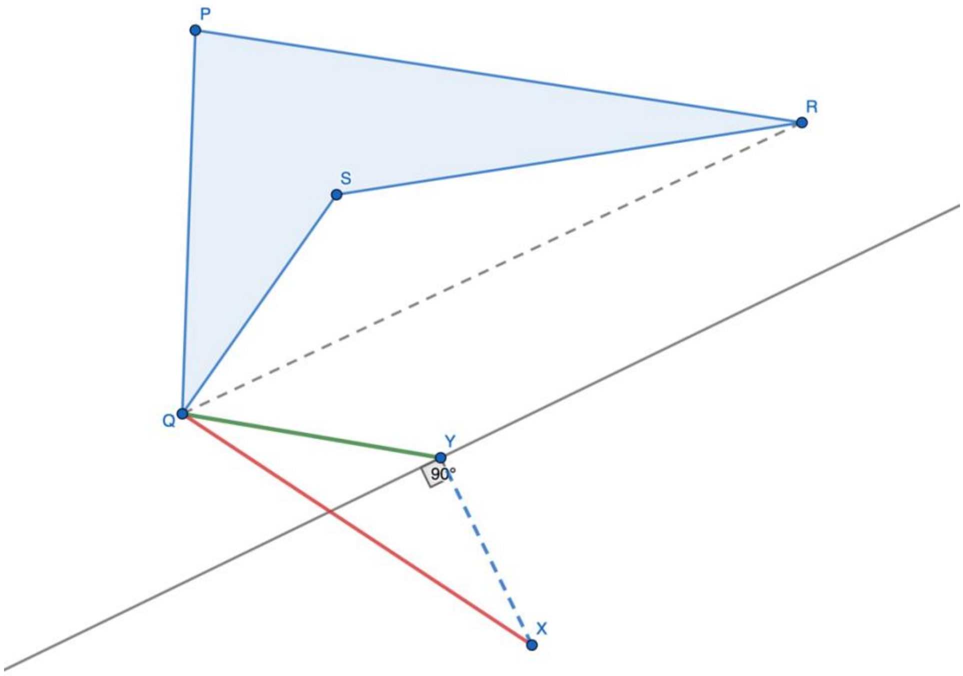

Figure 11: XQ is strictly greater than YQ.

Case 2: X in PQR

For any point X inside triangle PQR, S is a point inside PQX, P and Q are two vertices of an arbitrarily named PQR. Then, extend XS to meet PQ at a point Y. By triangle inequality,

$$\boldsymbol{QS \leq SY + YQ}$$

$$\boldsymbol{PS + QS \leq PS + YS + YQ = PY + QX - XY}$$

$$\boldsymbol{PY \leq PX + XY}$$

$$PY + QX - XY \leq PX + XY + YQ = PX + QX$$

Therefore,

$$PS + QS \leq PX + QX$$

$$RS \leq RX + SX$$

Adding up gives

$$PS + QS + RS \leq PX + QX + RX + XS$$

Since $SS = 0$ we have

$$PS + QS + RS + SS \leq PX + QX + RX + XS$$

As a result, S is the Fermat point for a concave quadrilateral.

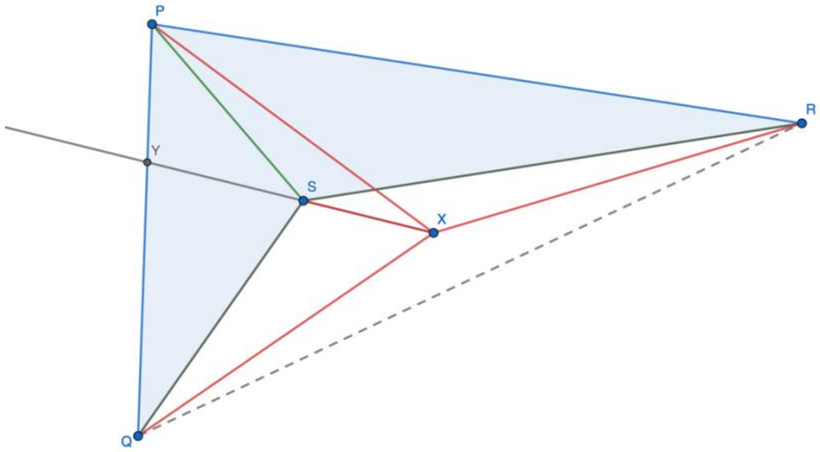

Figure 12: S is the Fermat Point for the concave case.

3.2 Algebraic method

Using the methods of Escobar-Villagram et. al, the general expression for a convex quadrilateral can be derived. We arrive at the two partial derivative equations that equate to 0:

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{4} \frac{x - a_i}{\sqrt{(x - a_i)^2 + (y - b_i)^2}} = 0$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{4} \frac{y - b_i}{\sqrt{(x - a_i)^2 + (y - b_i)^2}} = 0$$

However, expanding out the radicals do not simplify nicely to the 3 (in this case, 4) polynomial equations, and trying to expand out the roots yield a very large polynomial of degree 12. The dimension reduction tricks used by Escobar-Villagram were not applicable to the case of 4 points, and therefore it was concluded that isolating for x and y was not feasible.

Then, we tried to perform a few simplifications to the questions by restricting the quadrilateral to a parallelogram, trapezium, and a right-angled trapezium respectively. This sets a few conditions as illustrated below.

Parallelogram	Trapezium	Right-angled trapezium
$a_3 = a_2 + a_4$ $b_4 = b_3$ $a_1 = b_1 = b_2 = 0$	$b_4 = b_3$ $a_1 = b_1 = b_2 = 0$	$b_4 = b_3$ $a_1 = a_4 = b_1 = b_2 = 0$

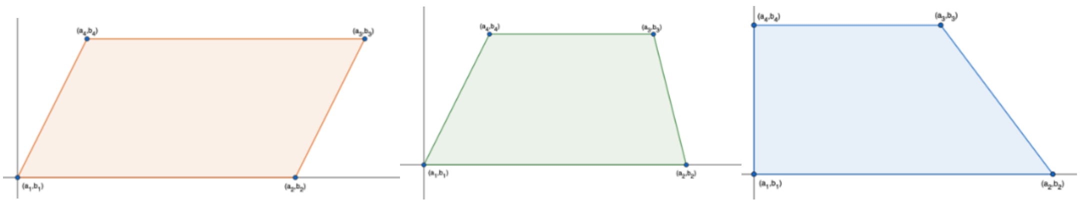

Figure 13: constructions of each quadrilateral given above constraints

However, even after substituting in the appropriate constraints, Wolfram Mathematica was unable to solve the equation after 2 hours.

However, this does not mean that an equation solution to the Fermat point cannot be obtained. By applying the results from the geometric solution (the fact that the Fermat point occurs at the intersection of the diagonals of the quadrilateral), one can obtain the equation for the floating case quite easily.

We start by defining the vertices of the convex quadrilateral in the same notation as Section 2.4:

$$(a_1,\ b_1), (a_2,\ b_2), (a_3,\ b_3), (a_4,\ b_4)$$

Using the same restrictions,

$$(a_1,\ b_1) = (0, 0) \qquad\qquad (a_2, b_2) = (a_2\ , 0)$$

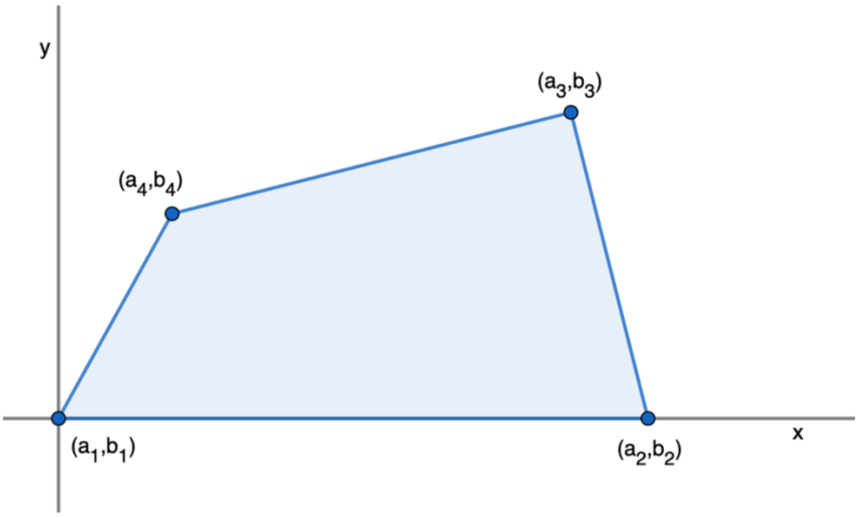

Figure 14: restricting two coordinates of a convex quadrilateral

First, obtain the linear equations of the two diagonals in point-slope form:

$$l_1: (y - 0) = \frac{b_3}{a_3}(x - 0)$$

$$l_2: (y - 0) = \frac{b_4}{a_4 - a_2}(x - a_2)$$

Next, equate the two together and isolate for x.[6]

$$\frac{b_3}{a_3}x = \frac{b_4}{a_4 - a_2}(x - a_2)$$

$$x = \frac{a_2 a_3 b_4}{a_2 b_3 - a_4 b_3 + a_3 b_4}$$

$$y = \frac{a_2 b_3 b_4}{a_2 b_3 - a_4 b_3 + a_3 b_4}$$

Substituting these for x and y in the original partial derivative equations and plugging in random coordinates for a convex quadrilateral, we find out that the derived equations agree with the original partial derivative equations (Fig. 15).

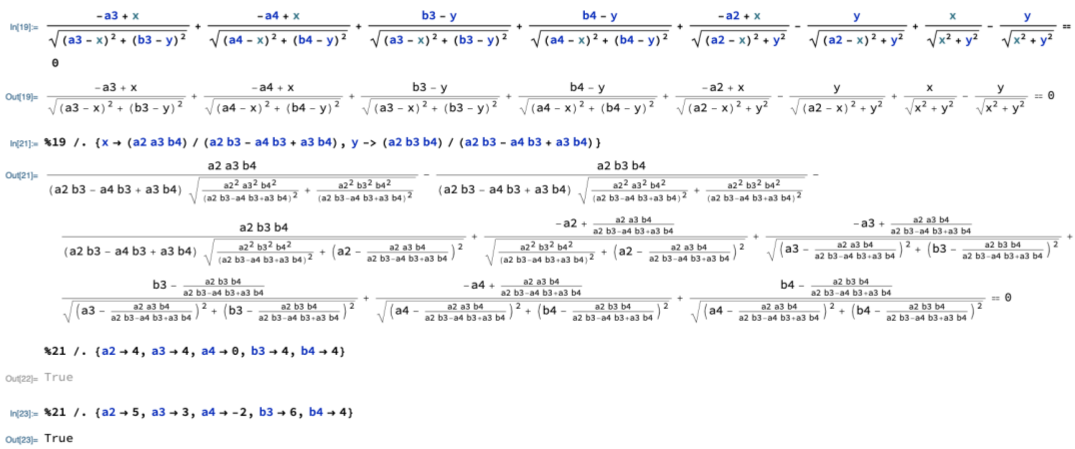

Figure 15: Plugged in coordinates for a square and a random quadrilateral. Mathematica returned true, showing that the results agree with the partial derivative equations.

As for the absorbed case, the solution is trivial. Since the Fermat Point is located at the concave vertex, the equations are merely

$$x = a_{vertex}$$

$$y = b_{vertex}$$

[6] Some steps are omitted for readability. Those can be found in the Appendix.

It must be noted that the equations for the floating case do not work for the absorbed case as the diagonals of a concave quadrilateral do not intersect with each other. If one were to extend the diagonal to make the two lines to intersect, that intersection is not the Fermat point, by a similar proof to the ≥120˚ case.

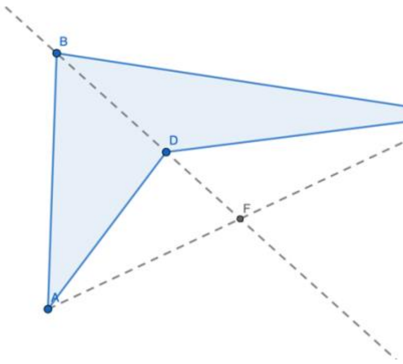

Figure 16: the diagonals of a concave quadrilateral do not intersect at the Fermat point.

4.0 Five Points and Beyond

4.1 Geometric Method

The Fermat point of 5 vertices is not constructible (Bajaj, 1998). In order to explain why it is not, we first consider the set of all points in a two-dimensional plane we can construct using an unmarked ruler and compass. Any potential geometric method using a ruler and compass can only return such points by definition. Therefore, **if we can show that the Fermat point for 5 vertices is not such a point, we have shown that we can never find a geometric construction for this point.**

4.1.1 Constructible numbers

To begin the section, a few definitions and scopes must first be clarified.

Definition: The **geometric construction** method referred to in this paper constitutes of an unmarked straight edge and a compass. That is, in a two-dimensional plane, a line can be drawn between any two points, and a circle can be drawn around any point. It ought to be noted that the radius of this circle is determined by the center of the circle and a point on the line that intersects with the circle.

Definition: We start by drawing two arbitrary points, labelled (0,0) and (1,0), drawing line L through them. A real number r is constructible iff there is a finite sequence of straightedge-compass actions that can be used to create a point P equal to r (Bartlett, 2014).

Draw a circle C around (1,0) that passes through (0,0), yielding two intersections: (0,0) and (P,0). By the properties of a circle, the distance from (0,0) to (P,0) is equal to twice radius of C, and since the distance from (0,0) to (P,0) is 1, P must be (1+1,0) = (2,0).

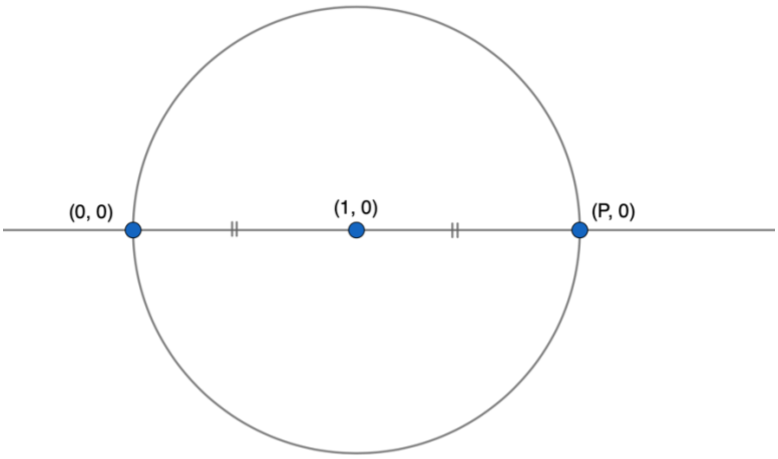

Figure 17: Construction of (2, 0).

This allows us to prove:

Theorem 1: All points in the form (k, 0) where k is an integer are constructible.

Proof:

We prove by induction. Having assumed that (0, 0) and (1, 0) are constructible, we now assume that (k,0) can be constructed for $1 \leq k \leq n$. Now, draw a circle C around (k,0) passing through (k-1, 0). As in the first example, C intersects L at a distance 2 from (k-1, 0), namely (k-1+2, 0) = (k+1, 0). Hence, (k+1, 0) can be constructed if (k, 0) can be constructed. Given that this premise is true for n = 1, k, and k+1, by the principle of mathematical induction, the premise is true for all positive integers.

Negative integers are formed by applying the same procedure in the opposite direction.

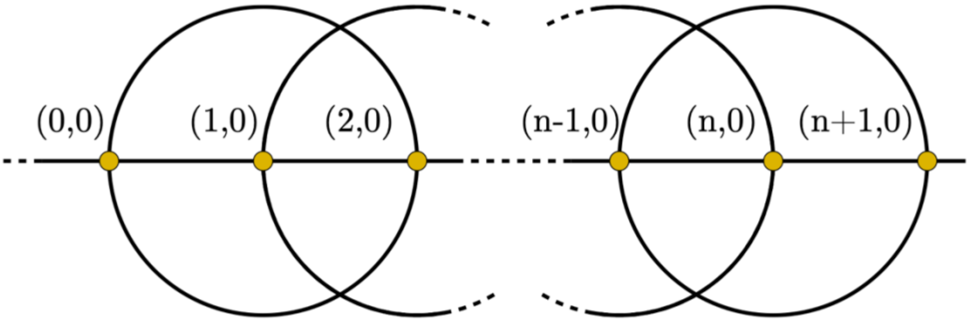

Figure 18: Visual representation of the proof by induction. Taken from Bartlett (2014).

As it turns out, there are a few things that we can construct geometrically:

1. Perpendicular lines.
2. Parallel lines.
3. If a and b are constructible, (a, b) is constructible.
4. If (a, b) and $x \neq 0$ are constructible, then so are (xa, xb) and (a/x, b/x).
5. Known distances can be transferred (also known as the "Non-collapsing Compass")

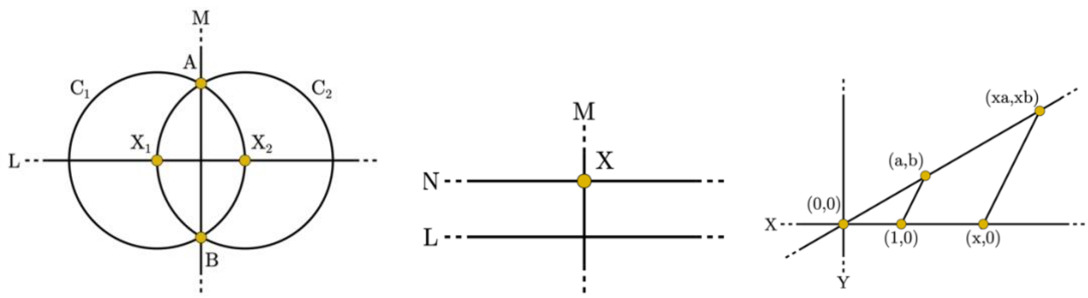

Figure 19: construction for perpendicular and parallel lines, as well as multiplying (a,b) by a constant. Taken from Bartlett (2014).

It also turns out that performing basic operations (ie $+ - \times \div$) on constructible numbers yield numbers that are constructible themselves (Bartlett, 2014). With the above in mind, a few types of numbers can be seen to be constructible.

1. As discussed earlier, **integers** are constructible.
2. Through division, **rationals with integer numerators and denominators** are constructible.
3. **Square roots** are constructible (Fig. 20)

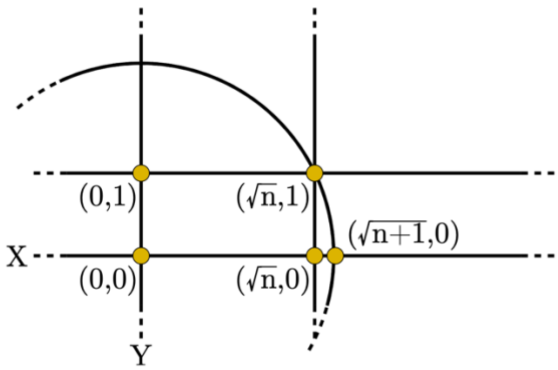

Figure 20: Inductive step for the construction of a square root. Taken from Bartlett (2014).

4.1.2 Unconstructible numbers

So far, a few types of numbers have been established to be constructible. But what kinds of numbers are not?

Cube roots are not constructible (Janson, 2009). This is an ancient problem known as "Doubling the Cube" that has remained an open problem for centuries before recent advances in abstract algebra. Additionally, in general, nth roots are not constructible (Wantzel, 1837).

Bajaj proves that $n \geq 5$, the situation is much more complicated. In this paper, they show that the Fermat point problem reduces to solving a polynomial equation. They then show that the roots of this equations often are non-constructible. Since the coordinates of such a Fermat point is unable to be constructed using a straightedge and a ruler, **a geometric solution for the Fermat point beyond four vertices is generally impossible.**

4.2 Algebraic method?

Since there are no tricks that can be exploited like for the case of 4 points, it does not seem possible to isolate for the coordinates of the Fermat point given current methods. With reference to Bajaj (1998), **seeking an algebraic approach to a high degree polynomial equation is unfeasible.**

Since it is not possible to do this algebraically, we follow with a brief discussion on a few numerical methods that could theoretically be used to tackle the problem.

4.3 Numerical methods

As suggested in 3.2, the polynomial equation of high degree. Since this is an equation of two unknowns, a numerical method could be used to find approximations. One such method is called the Runge-Kutta method. This method uses the information of the derivatives of the equation at more than one point to extrapolate the solution of the next step (Zeltkevic, 1998). The method for calculating some y_{n+1} given y_n is shown below:

$$k_1 = hf(y_n, t_n)$$

$$k_2 = hf(y_n + \frac{k_1}{2}, t_n + \frac{h}{2})$$

$$k_3 = hf(y_n + \frac{k_2}{2}, t_n + \frac{h}{2})$$

$$k_4 = hf(y_n + k_3, t_n + h)$$

$$y_{n+1} = y_n + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

Where h is an arbitrarily chosen time step and $f(y_n, t_n)$ is the polynomial equation.

A solution can be approximated by machine learning algorithms. Consider $f(x, y)$ as a hilly terrain with the height as cost (aka distance). An algorithm such as SVM can follow the gradients to arrive at the troughs in the terrain (aka local minimums, which happens to be a global minimum in this problem) (Simons, 2003).

5.0 Conclusion

Is an analytical solution for the generalized Fermat Point possible? Not geometrically or algebraically given the methods investigated in this paper.

In this paper, we discussed an elegant geometric method to finding the Fermat Point discovered centuries ago, and a more complicated algebraic method. We find out that such a Fermat point is unique and forms 120° angles with the triangle vertices.

Finding the Fermat Point for 4 vertices proved to be quite simple as well, with a proof by triangle inequality, it could be seen that this Fermat point is the intersection of the diagonals of a quadrilateral. However, trying to find isolated algebraic equations for this Fermat point proved extremely difficult using the algebraic approach, and could only be done using the abovementioned geometric property.

Trying to generalize the Fermat point for 5 vertices or more proved to be difficult. It can be concluded that the generalized Fermat point cannot be constructed geometrically and attempting to investigate the algebraic approach was beyond the scope of this paper.

Only the most common geometric and algebraic methods were explored in this paper. Perhaps there are ways to arrive at an analytical solution to the Fermat Point via such methods, which is a possible direction for further research.

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Appendix 1: Calculations for 2.4

Let:

$$x - a_1 = A \qquad y - b_1 = a$$

$$x - a_2 = B \qquad y - b_2 = b$$

$$x - a_3 = C \qquad y - b_3 = c$$

$$\left(\frac{A}{\sqrt{A^2 + a^2}} + \frac{B}{\sqrt{B^2 + b^2}} + \frac{C}{\sqrt{C^2 + c^2}} = 0\right) \times (A^2 + a^2)(B^2 + b^2)(C^2 + c^2)$$

$$\left(\frac{a}{\sqrt{A^2 + a^2}} + \frac{b}{\sqrt{B^2 + b^2}} + \frac{c}{\sqrt{C^2 + c^2}} = 0\right) \times (A^2 + a^2)(B^2 + b^2)(C^2 + c^2)$$

$$A\sqrt{A^2 + a^2}(B^2 + b^2)(C^2 + c^2) + B\sqrt{B^2 + b^2}(A^2 + a^2)(C^2 + c^2) + C\sqrt{C^2 + c^2}(A^2 + a^2)(B^2 + b^2) = 0$$

$$a\sqrt{A^2 + a^2}(B^2 + b^2)(C^2 + c^2) + b\sqrt{B^2 + b^2}(A^2 + a^2)(C^2 + c^2) + c\sqrt{C^2 + c^2}(A^2 + a^2)(B^2 + b^2) = 0$$

$$\sqrt{A^2 + a^2}(B^2 + b^2)(C^2 + c^2) = \frac{-B\sqrt{B^2 + b^2}(A^2 + a^2)(C^2 + c^2) - C\sqrt{C^2 + c^2}(A^2 + a^2)(B^2 + b^2)}{A} = \frac{-b\sqrt{B^2 + b^2}(A^2 + a^2)(C^2 + c^2) - c\sqrt{C^2 + c^2}(A^2 + a^2)(B^2 + b^2)}{a}$$

$$aB(C^2 + c^2)\sqrt{B^2 + b^2} + aC(B^2 + b^2)\sqrt{C^2 + c^2} = Ab(C^2 + c^2)\sqrt{B^2 + b^2} + Ac(B^2 + b^2)\sqrt{C^2 + c^2}$$

$$[aB - Ab](C^2 + c^2)\sqrt{B^2 + b^2} = [Ac - aC](B^2 + b^2)\sqrt{C^2 + c^2}$$

$$\frac{(aB - Ab)^2}{(Ac - aC)^2} = \frac{B^2 + b^2}{C^2 + c^2}$$

$$\frac{((x - a_2)(y - b_1) - (x - a_1)(y - b_2))^2}{((x - a_3)(y - b_1) - (x - a_1)(y - b_3))^2} = \frac{(x - a_2)^2 + (y - b_2)^2}{(x - a_3)^2 + (y - b_3)^2}$$

The other two equations can be obtained by isolating $\sqrt{B^2 + b^2}(A^2 + a^2)(C^2 + c^2)$ and $\sqrt{C^2 + c^2}(A^2 + a^2)(B^2 + b^2)$.

Appendix 2: Raw code for 2.4.1

```
import numpy as np
import math
from tqdm import tqdm

#cosine law function
def angle (a, b, c):
        return math.degrees(math.acos(round((c**2 - b**2 - a**2)/(-2.0 * a * b),6)))

#check if angles less than 120˚ function
def check (a2, a3, b3):
        a = a2
        b = math.sqrt((a2-a3)**2+b3**2)
        c = math.sqrt(a3**2+b3**2)

        angA = angle(a,b,c)
        angB = angle(b,c,a)
        angC = angle(c,a,b)

        if round(angA + angB + angC,2) != 180.00:
                return False
        elif angA > 120.0 or angB > 120.0 or angC > 120.0:
                return False
        else:
                return True



#plug into the authors' formulas
def fermatpoint (a2, a3, b3):
        x = a2*(math.sqrt(3)*a3 + b3)*(a2+a3+math.sqrt(3)*b3)/(2*math.sqrt(3)*(a2**2-
a2*a3+a3**2+b3**2+math.sqrt(3)*a2*b3))
        y = a2*(math.sqrt(3)*a3 + b3)*(math.sqrt(3)*(a2-a3)+b3)/(2*math.sqrt(3)*(a2**2-
a2*a3+a3**2+b3**2+math.sqrt(3)*a2*b3))
        return (x,y)

def fermatpointtrue(n):
        for i in tqdm(range(n)):
```

```
        a2 = np.random.randint(1,1000000000000)
        a3 = np.random.randint(1,1000000000000)
        b3 = np.random.randint(1,1000000000000)
        check(a2, a3, b3)

        #generates a triangle with interior angles no more than 120˚
        while check(a2, a3, b3) == False:
            a2 = np.random.randint(1,1000000000000)
            a3 = np.random.randint(1,1000000000000)
            b3 = np.random.randint(1,1000000000000)
            check(a2, a3, b3)

        #define side lengths.
        a = a2
        b = math.sqrt((a2-a3)**2+b3**2)
        c = math.sqrt(a3**2+b3**2)

        #grab coordinates of fermat point
        x = fermatpoint(a2, a3, b3)[0]
        y = fermatpoint(a2, a3, b3)[1]

        #determine lengths of the segment joining the fermat point to each of the vertices.
        P1 = math.sqrt(x**2+y**2)
        P2 = math.sqrt((x-a2)**2+y**2)
        P3 = math.sqrt((x-a3)**2+(y-b3)**2)

        #find the partials
        pdx = round(x/P1 + (x-a2)/P2 + (x-a3)/P3, 2)
        pdy = round(y/P1 + y/P2 + (y-b3)/P3, 2)

        #check if the partials equal 0. If not, (x,y) is not the Fermat Point and print the properties of such a
configuration.
        if pdx != 0 or pdy != 0:
            print("number of iterations: ",n)
            print(' ')
            print("original coords: ", a2, a3, b3)
            print('')
            print("fermat point coords: ", x, y)
```

```
            print('')
            print("side lengths: ", a, b, c)
            print('')
            print("P1, P2, P3: ", P1, P2, P3)
            print("pdx, pdy = ", pdx, pdy)
            return False


fermatpointtrue(100000000)
```

Nothing was printed, meaning all samples satisfied the condition for the partial derivatives to equal 0.

The code for triangles with an interior angle of 120˚ and greater than 120˚ respectively is the same as above but with slight modifications to the triangle generation.

Appendix 3: Raw calculations for 3.2

$$l_1: (y - 0) = \frac{b_3}{a_3}(x - 0)$$

$$l_2: (y - 0) = \frac{b_4}{a_4 - a_2}(x - a_2)$$

$$\frac{b_3}{a_3}x = \frac{b_4}{a_4 - a_2}(x - a_2)$$

$$\frac{b_3}{a_3}x = \frac{b_4}{a_4 - a_2}x - \frac{b_4}{a_4 - a_2}a_2$$

$$\frac{b_4 a_2}{a_4 - a_2} = \frac{b_4}{a_4 - a_2}x - \frac{b_3}{a_3}x$$

$$\left(\frac{b_4}{a_4 - a_2} - \frac{b_3}{a_3}\right)x = \frac{b_4 a_2}{a_4 - a_2}$$

$$x = \frac{\frac{b_4 a_2}{a_4 - a_2}}{\frac{b_4}{a_4 - a_2} - \frac{b_3}{a_3}}$$

$$= \frac{\frac{b_4 a_2}{a_4 - a_2}}{\frac{b_4}{a_4 - a_2} \times \frac{a_3}{a_3} - \frac{b_3}{a_3} \times \frac{a_4 - a_2}{a_4 - a_2}}$$

$$= \frac{\frac{b_4 a_2}{a_4 - a_2}}{\frac{b_4 a_3}{a_3(a_4 - a_2)} - \frac{b_3 a_4 - b_3 a_2}{a_3(a_4 - a_2)}}$$

$$= \frac{\frac{b_4 a_2}{a_4 - a_2}}{\frac{b_4 a_3 - b_3 a_4 + b_3 a_2}{a_3(a_4 - a_2)}}$$

$$= \frac{a_2 a_3 b_4}{a_2 b_3 - a_4 b_3 + a_3 b_4}$$

$$y = \frac{b_3}{a_3}x$$

$$= \frac{b_3}{a_3} \times \frac{a_2 a_3 b_4}{a_2 b_3 - a_4 b_3 + a_3 b_4}$$

$$= \frac{a_2 b_3 b_4}{a_2 b_3 - a_4 b_3 + a_3 b_4}$$